ELEC0029 - Electric power systems analysis

The power flow computation

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English terminology:

- *load flow* computation
- *power flow* computation (preferred)

Termes français :

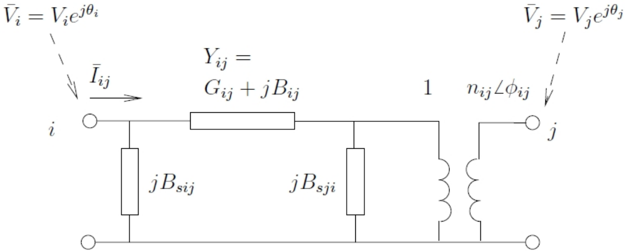
- calcul de *répartition de charge*
- calcul *d'écoulement de charge*
- calcul *d'écoulement de puissance*.

Objective: determine the electrical state of a network (voltages at all buses, currents in all branches, losses, etc.) from consumptions and productions specified at its various buses.

Certainly the most used computation in electric power systems !

Power flow equations

"Multi-purpose" two-port

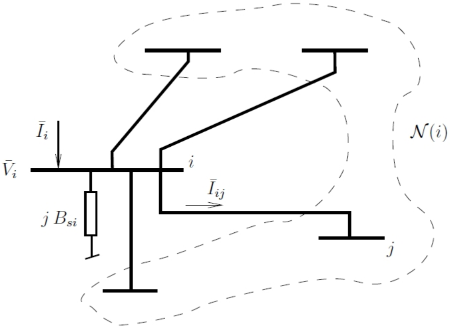


To model:

- a line or a cable: set $n_{ij} = 1$ and $\phi_{ij} = 0$
- a standard transformer: set $B_{sji} = 0$ and $\phi_{ij} = 0$
- a phase-shifting transformer: set $B_{sji} = 0$ and $\phi_{ij} \neq 0$

$$\bar{I}_{ij} = jB_{sij}\bar{V}_i + Y_{ij}(\bar{V}_i - \frac{\bar{V}_j}{n_{ij}}e^{-j\phi_{ij}}) = jB_{sij}\bar{V}_i + (G_{ij} + jB_{ij})(\bar{V}_i - \frac{\bar{V}_j}{n_{ij}}e^{-j\phi_{ij}})$$

Power balances at the network buses



$$\bar{I}_i \;=\; j\,B_{si}\,\bar{V}_i + \sum_{j \in \mathcal{N}(i)} \bar{I}_{ij} \qquad\qquad i = 1, \ldots, N$$

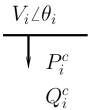
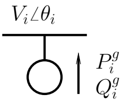
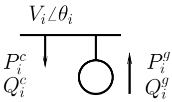
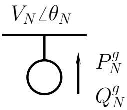
$$\Leftrightarrow \quad P_i + jQ_i = \bar{V}_i\,\bar{I}_i^{\star} \;=\; -jB_{si}\,V_i^2 + \sum_{j \in \mathcal{N}(i)} \bar{V}_i\,\bar{I}_{ij}^{\star} \qquad\qquad i = 1, \ldots, N$$

$$
\begin{aligned}
P_i + jQ_i &= -jB_{si}V_i^2 + \sum_{j\in\mathcal{N}(i)} \bar{V}_i \bar{I}_{ij}^\star \\
&= -jB_{si}V_i^2 - j\sum_{j\in\mathcal{N}(i)} B_{sij}V_i^2 + \sum_{j\in\mathcal{N}(i)} \bar{V}_i(G_{ij}-jB_{ij})(V_i^\star - \frac{\bar{V}_j^\star}{n_{ij}}e^{j\phi_{ij}}) \\
&= -jB_{si}V_i^2 - j\sum_{j\in\mathcal{N}(i)} B_{sij}V_i^2 + \sum_{j\in\mathcal{N}(i)} (G_{ij}-jB_{ij})(V_i^2 - \frac{V_iV_j}{n_{ij}}e^{j(\theta_i-\theta_j+\phi_{ij})})
\end{aligned}
$$

$$
P_i = \underbrace{\sum_{j\in\mathcal{N}(i)} G_{ij}V_i^2 - \sum_{j\in\mathcal{N}(i)} \frac{V_iV_j}{n_{ij}}\left[G_{ij}\cos(\theta_i-\theta_j+\phi_{ij}) + B_{ij}\sin(\theta_i-\theta_j+\phi_{ij})\right]}_{f_i(\ldots,V_i,\theta_i,\ldots)}
$$

$$
Q_i = \underbrace{-(B_{si} + \sum_{j\in\mathcal{N}(i)} B_{sij} + \sum_{j\in\mathcal{N}(i)} B_{ij})V_i^2 + \sum_{j\in\mathcal{N}(i)} \frac{V_iV_j}{n_{ij}}\left[B_{ij}\cos(\theta_i-\theta_j+\phi_{ij}) - G_{ij}\sin(\theta_i-\theta_j+\phi_{ij})\right]}_{g_i(\ldots,V_i,\theta_i,\ldots)}
$$

Data specification

LOAD BUS	GENERATOR BUS		LOAD & GENERATOR BUS		SLACK BUS
					
EQUATIONS					
$-P_i^c = f_i(\dots)$	$P_i^g = f_i(\dots)$		$P_i^g - P_i^c = f_i(\dots)$		$\theta_N = 0$
$-Q_i^c = g_i(\dots)$	$Q_i^g = g_i(\dots)$	$V_i = V_i^o$	$Q_i^g - Q_i^c = g_i(\dots)$	$V_i = V_i^o$	$V_N = V_N^o$
PQ BUS	**PQ BUS**	**PV BUS**	**PQ BUS**	**PV BUS**	**$V\theta$ BUS**
UNKNOWNS					
V_i θ_i	V_i θ_i	θ_i	V_i θ_i	θ_i	
by substituting voltage magnitudes and phase angles in the non used power balance equation, one obtains:					
		Q_i^g		Q_i^g	P_N^g Q_N^g

The role of the slack bus

1.

One cannot specify the active powers injected P_i at *all* buses since:

$$\sum_{i=1}^{N} P_i = \underbrace{\text{active losses} = p\,(\ldots, V_i, \theta_i, \ldots)}_{??}$$

2.

- the voltage phase angles θ_i appear in the eqs. through *differences only*
- one may add an arbitrary constant c to all phase angles without changing the electric state of the network
- one bus must be taken as phase angle *reference*.

Slack bus: let us assume it is the N-th bus:

active power balance equation replaced by:

$$\theta_N = 0 \qquad\qquad (0 : \text{ arbitrary value})$$

Which bus take as slack bus?

At this bus, the active power injection will take the value:

$$P_N = -\sum_{i=1}^{N-1} P_i + p$$

where all P_i's in the sum are data and p is known at the end of the computation

$\Rightarrow$ select a bus to which a generator is connected (= *slack generator*)

What about reactive losses and the role of the slack bus?

- one cannot specify the reactive power injections at *all* buses
- luckily, they are not imposed at the PV buses !
- $\Rightarrow$ no problem as long as there are one or more PV buses in the data (for reactive power, each PV bus acts as a slack bus).

Which data specify at the slack bus?

- either the voltage V or the reactive power injection Q
- since a generator is connected, it is natural to specify the voltage V.

Power flow equations in vector form

N_{PV} : nb of PV buses $\quad$ N_{PQ} : nb of PQ buses $\quad$ $N_{PV} + N_{PQ} + 1 = N$

$\boldsymbol{v}$ vector of voltage magnitudes at all buses (dim. N)
$\boldsymbol{\theta}$ vector of voltage phase angles at all buses (dim. N)
$\boldsymbol{p}^o$ vector of specified active power injections at PV or PQ buses (dim. $N-1$)
$\mathbf{q}^o$ vector of specified reactive power injections at PQ buses (dim. N_{PQ})
$\boldsymbol{v}^o$ vector of specified voltage magnitudes at PV and slack buses (dim $N_{PV}+1$)

$N-1$ active power balance equations at PV and PQ buses:

$$\boldsymbol{f}(\boldsymbol{v},\boldsymbol{\theta}) - \boldsymbol{p}^o = \mathbf{0}$$

1 phase angle equation at the slack bus:

$$\theta_N = 0$$

N_{PQ} reactive power balance equations at PQ buses:

$$\boldsymbol{g}(\boldsymbol{v},\boldsymbol{\theta}) - \boldsymbol{q}^o = \mathbf{0}$$

$N_{PV}+1$ voltage equations at the PV and slack buses:

$$\boldsymbol{v} - \boldsymbol{v}^o = \mathbf{0}$$

Solving the equations numerically

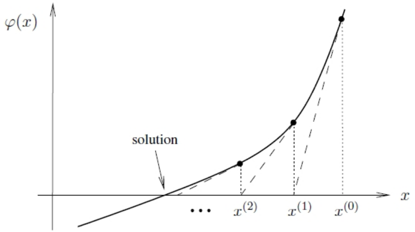
Newton(-Raphson) method: case of a scalar function of a single variable

Let the equation be: $\varphi(x) = 0.$ We note $\varphi_x = \dfrac{d\varphi}{dx}.$

Starting from an initial value ("guess") $x^{(0)}$, the following sequence is computed:

$$x^{(k+1)} = x^{(k)} - \frac{\varphi(x^{(k)})}{\varphi_x(x^{(k)})} \qquad k = 0, 1, 2, \ldots$$

until: $|\varphi(x^{(k+1)})| < \epsilon$



For $x^{(0)}$ "sufficiently close" to the solution, the convergence is fast (quadratic).

Newton method: case of a vector function of several variables

Let the equation be: $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{0}$

The *Jacobian (matrix)* of $\boldsymbol{\varphi}$ with respect to $\boldsymbol{x}$ is defined by:

$$[\boldsymbol{\varphi}_x]_{ij} = \frac{\partial \varphi_i}{\partial x_j}$$

Starting from an initial value ("guess") $\boldsymbol{x}^{(0)}$, the following sequence is computed:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - [\boldsymbol{\varphi}_x(\boldsymbol{x}^{(k)})]^{-1}\boldsymbol{\varphi}(\boldsymbol{x}^{(k)}) \qquad k = 0, 1, 2, \ldots$$

until: $\max_i |\varphi_i(\boldsymbol{x}^{(k+1)})| < \epsilon$.

In pratice, the Jacobian is not inverted. Instead, the linear system:

$$\boldsymbol{\varphi}_x(\boldsymbol{x}^{(k)})\, \Delta\boldsymbol{x} = -\boldsymbol{\varphi}(\boldsymbol{x}^{(k)})$$

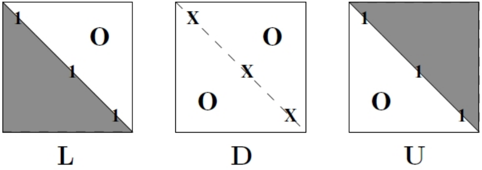
is solved and $\boldsymbol{x}$ is incremented according to:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \Delta\boldsymbol{x}$$

Solving the linear system

Factorization (or *LDU decomposition*) of the Jacobian:

$$\varphi_x = \boldsymbol{LDU}$$



Substitution of the right-hand side term:

$$\varphi_x \Delta \boldsymbol{x} = -\varphi \qquad \Leftrightarrow \qquad \boldsymbol{LDU}\Delta \boldsymbol{x} = -\varphi$$

one solves successively:

$$\begin{aligned} \boldsymbol{Ly} &= -\varphi \\ \boldsymbol{U}\Delta \boldsymbol{x} &= \boldsymbol{D}^{-1}\boldsymbol{y} \end{aligned}$$

which is easy owing to the triangular structure of the matrices.

Application to power flow equations

Structure of Jacobian:

$$\varphi_x = \left[\begin{array}{c|c} \boldsymbol{f_v} & \boldsymbol{f_\theta} \\ \mathbf{0} & \boldsymbol{e_N} \\ \hline \boldsymbol{g_v} & \boldsymbol{g_\theta} \\ \boldsymbol{U} & \mathbf{0} \end{array}\right]$$

$\boldsymbol{e_N}$: unit row vector of dimension N
$\boldsymbol{U}$: matrix whose entries are 0's and 1's
Each of the four sub-matrices of dimension $N \times N$.

The matrix φ_x is *sparse*: most of the elements are zero, since a power injection at one bus involves only the voltages at this bus and at its direct neighbours.

- only the nonzero terms are stored, in a compact way (accessed through pointers)
- useless operations involving zero's are avoided (*sparsity programming*)
- the number of nonzero terms created during the factorization (*fill-in terms*) is kept as low as possible, by permuting rows and columns: *optimal ordering*
- the pattern of nonzero terms is analyzed, often before the factorization step.

Another saving of computing time:

- the Jacobian $\varphi_{\mathbf{x}}$ is no longer updated when $\varphi(\boldsymbol{x})$ has become small enough (i.e. when $\boldsymbol{x}$ is close enough to the solution)
- only the substitution of the right-hand side $-\varphi(\boldsymbol{x}^{(k)})$ is performed, reusing the available $\boldsymbol{L}$, $\boldsymbol{D}$ et $\boldsymbol{U}$ matrices
- provided the iterations converge, the obtained solution does not depend on matrix $\varphi_{\mathbf{x}}$.

Initial value $\boldsymbol{x}^{(0)}$: if a more precise estimate is not available:

- at each PV and at the slack bus: voltage magnitude set to the value specified in the data
- at other buses: voltage magnitude set to 1 pu
- at all buses: voltage phase angle set to zero ($=$ the phase angle imposed at the slack bus).

Taking operating constraints into account

If the solution does not obey an operating constraint:

$$h(\boldsymbol{v}, \boldsymbol{\theta}) \leq 0$$

one can enforce:

$$h(\boldsymbol{v}, \boldsymbol{\theta}) = 0$$

provided that another equation is removed.

The resulting new set of equations is solved. The procedure is repeated if needed.

Important case : reactive power limits of generators at PV buses

$$Q_i^{min} \leq Q_i(\boldsymbol{v}, \boldsymbol{\theta}) \leq Q_i^{max}$$

- if $Q_i(\boldsymbol{v}, \boldsymbol{\theta}) > Q_i^{max}$: the bus is switched from PV to PQ type, with $Q_i = Q_i^{max}$
  V_i becomes variable: $V_i = V_i^o \quad \rightarrow \quad V_i \leq V_i^o$
- if $Q_i(\boldsymbol{v}, \boldsymbol{\theta}) < Q_i^{min}$: the bus is switched from PV to PQ type, with $Q_i = Q_i^{min}$
  V_i becomes variable: $V_i = V_i^o \quad \rightarrow \quad V_i \geq V_i^o$.

Newton method with enforcement of generator reactive power limits

1. $k := 0$; $\boldsymbol{v}^{(0)} = \boldsymbol{v}^o$; $\boldsymbol{\theta}^{(0)} = \boldsymbol{\theta}^o$
2. compute $\boldsymbol{f}(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - \mathbf{p}^o$ and $\boldsymbol{g}(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - \mathbf{q}^o$
3. **if** $\max_i |g_i(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - Q_i^o| < \delta_Q$:
   check generator reactive power limits ;
   if some limits exceeded :
   switch the corresponding buses from PV to PQ
   go to 2 ;
4. **if** $\max_i |f_i(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - P_i^o| < \epsilon_P$ and $\max_i |g_i(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - Q_i^o| < \epsilon_Q$:
   stop ;
5. **if** $k = 0$ or $\max_i |f_i(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - P_i^o| > \beta_P$ or $\max_i |g_i(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) - Q_i^o| > \beta_Q$:
   compute and factorize the Jacobian: $\varphi_{\boldsymbol{x}} = \boldsymbol{LDU}$
6. (substitution:) solve $\boldsymbol{LDU} \begin{bmatrix} \Delta \boldsymbol{v} \\ \Delta \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^o - \boldsymbol{f}(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) \\ \mathbf{q}^o - \boldsymbol{g}(\boldsymbol{v}^{(k)}, \boldsymbol{\theta}^{(k)}) \end{bmatrix}$
7. $\boldsymbol{v}^{(k+1)} = \boldsymbol{v}^{(k)} + \Delta \boldsymbol{v}$; $\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \Delta \boldsymbol{\theta}$
8. $k := k + 1$; **go to** 2.

Iterative processing of losses

1. Estimate the network (active power) losses. Let us denote them by $\hat{p}$.
2. Share the sum of all loads and losses among the various generators, including the slack generator.

   Estimated slack generator production: $\hat{P}_N = -\sum_{i=1}^{N-1} P_i + \hat{p}$.
3. Run the power flow computation. This provides updated values of the losses p and the slack generator production P_N. Both are linked by:

   $$P_N = -\sum_{i=1}^{N-1} P_i + p$$

   and hence:

   $$P_N - \hat{P}_N = p - \hat{p}$$
4. If $|P_N - \hat{P}_N|$ is too large:

   adjust again the productions, using p as the new estimate of the losses

   go to 3.

   Else, stop.

In practice, no more than one iteration needed (i.e. 2 power flow computations).

Distributed slack buses

Share active power balancing among several generators (not just one)

$$\longrightarrow \textit{distributed slack buses}$$

Let us define an additional variable ΔP_g used to adjust the generations:

- at the i-th bus ($i = 1, \ldots, N$) a fraction α_i of ΔP_g is assigned:

$$P_i = P_i^o + \alpha_i \Delta P_g \qquad \text{with} \qquad \sum_{i=1}^{N} \alpha_i = 1$$

- if the i-th bus has no generator or has a generator not taking part in balancing:

$$\alpha_i = 0$$

The vector of bus active power injections can be written as:

$$\boldsymbol{p} = \boldsymbol{p}^o + \Delta P_g \, \boldsymbol{\alpha} \qquad \text{with} \qquad \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$$

Modified power flow equations:

- at all N buses:

$$\mathbf{f}(\boldsymbol{v}, \boldsymbol{\theta}) - \mathbf{p}^o - \Delta P_g \, \boldsymbol{\alpha} = \mathbf{0}$$

- at the phase angle reference bus:

$$\theta_N = 0$$

- at buses where reactive powers are specified:

$$\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{\theta}) - \boldsymbol{q}^o = \mathbf{0}$$

- at buses where voltage magnitudes are specified:

$$\boldsymbol{v} - \boldsymbol{v}^o = \mathbf{0}$$

$=$ $2N + 1$ equations with $2N + 1$ unknowns.

ΔP_g is computed (by Newton method) together with $\boldsymbol{v}$ and $\boldsymbol{\theta}$.

Adjusting the transformer ratios

Why?

- To keep the voltages V at some buses at specified values V^o
- to reflect an operating practice (operator adjusting transformers from a control center) or an automatic control device (load tap changer).

How?

- Ratios treated as continuous variables (approximation) or as discrete ones (corresponding to the tap positions)
- variation in constant steps or proportional to the deviation of voltage with respect to its set-point ($|V - V^o|$)
- taking in account the range of variation of each ratio (min/max tap positions)

- after the ratios have been adjusted, a new sequence of Newton iterations is performed
- the procedure is repeated until no ratio changes any further.

Electrical decoupling

$$P_i = \sum_{j \in \mathcal{N}(i)} G_{ij} V_i^2 - \sum_{j \in \mathcal{N}(i)} \frac{V_i V_j}{n_{ij}} \left[G_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) + B_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \right]$$

$$\frac{\partial P_i}{\partial V_i} = 2 V_i \sum_{j \in \mathcal{N}(i)} G_{ij} - \sum_{j \in \mathcal{N}(i)} \frac{V_j}{n_{ij}} \left[G_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) + B_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \right]$$

$$\frac{\partial P_i}{\partial V_j} = -\frac{V_i}{n_{ij}} \left[G_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) + B_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \right]$$

$$\frac{\partial P_i}{\partial \theta_i} = \sum_{j \in \mathcal{N}(i)} \frac{V_i V_j}{n_{ij}} \left[G_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) - B_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) \right]$$

$$\frac{\partial P_i}{\partial \theta_j} = -\frac{V_i V_j}{n_{ij}} \left[G_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) - B_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) \right]$$

Assuming that $V_i = V_j = n_{ij} \simeq 1$ pu and $\theta_i - \theta_j + \phi_{ij} \simeq 0$:

$$\frac{\partial P_i}{\partial V_i} \simeq \sum_{j \in \mathcal{N}(i)} G_{ij} \qquad \frac{\partial P_i}{\partial V_j} \simeq -G_{ij} \qquad \frac{\partial P_i}{\partial \theta_i} \simeq -\sum_{j \in \mathcal{N}(i)} B_{ij} \qquad \frac{\partial P_i}{\partial \theta_j} \simeq B_{ij}$$

$$G_{ij} \ll |B_{ij}| \quad \Rightarrow \quad \left|\frac{\partial P_i}{\partial V_i}\right|, \left|\frac{\partial P_i}{\partial V_j}\right| \ll \left|\frac{\partial P_i}{\partial \theta_i}\right|, \left|\frac{\partial P_i}{\partial \theta_j}\right|$$

$$Q_i = -[B_{si} + \sum_{j \in \mathcal{N}(i)} (B_{sij} + B_{ij})] V_i^2 + \sum_{j \in \mathcal{N}(i)} \frac{V_i V_j}{n_{ij}} [B_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) - G_{ij} \sin(\theta_i - \theta_j +$$

$$\frac{\partial Q_i}{\partial V_i} = -2[B_{si} + \sum_j (B_{sij} + B_{ij})] V_i + \sum_j \frac{V_j}{n_{ij}} [B_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) - G_{ij} \sin(\theta_i - \theta$$

$$\frac{\partial Q_i}{\partial V_j} = \frac{V_i}{n_{ij}} [B_{ij} \cos(\theta_i - \theta_j + \phi_{ij}) - G_{ij} \sin(\theta_i - \theta_j + \phi_{ij})]$$

$$\frac{\partial Q_i}{\partial \theta_i} = - \sum_{j \in \mathcal{N}(i)} \frac{V_i V_j}{n_{ij}} [B_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) + G_{ij} \cos(\theta_i - \theta_j + \phi_{ij})]$$

$$\frac{\partial Q_i}{\partial \theta_j} = \frac{V_i V_j}{n_{ij}} [B_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) + G_{ij} \cos(\theta_i - \theta_j + \phi_{ij})]$$

Assuming that $V_i = V_j = n_{ij} \simeq 1$ pu and $\theta_i - \theta_j + \phi_{ij} \simeq 0$:

$$\frac{\partial Q_i}{\partial V_i} \simeq -2[B_{si} + \sum_j B_{sij}] - \sum_j B_{ij} \qquad \frac{\partial Q_i}{\partial V_j} \simeq B_{ij} \qquad \frac{\partial Q_i}{\partial \theta_i} = -\sum_j G_{ij} \qquad \frac{\partial Q_i}{\partial \theta_j} \simeq G_{ij}$$

$$G_{ij} \ll |B_{ij}| \quad \Rightarrow \quad |\frac{\partial Q_i}{\partial V_i}|, \ |\frac{\partial Q_i}{\partial V_j}| \gg |\frac{\partial Q_i}{\partial \theta_i}|, \ |\frac{\partial Q_i}{\partial \theta_j}|$$

Jacobian matrix:

$$\text{full:}\quad \varphi_x = \left[\begin{array}{c|c} \boldsymbol{f_v} & \boldsymbol{f_\theta} \\ \mathbf{0} & \boldsymbol{e_N} \\ \hline \boldsymbol{g_v} & \boldsymbol{g_\theta} \\ \boldsymbol{U} & \mathbf{0} \end{array}\right] \qquad \text{approximate:}\quad \varphi_x \simeq \left[\begin{array}{c|c} \mathbf{0} & \boldsymbol{f_\theta} \\ \mathbf{0} & \boldsymbol{e_N} \\ \hline \boldsymbol{g_v} & \mathbf{0} \\ \boldsymbol{U} & \mathbf{0} \end{array}\right]$$

$\mathbf{f}_\theta$ dominant compared to $\boldsymbol{f_v}$ $\qquad$ $\boldsymbol{g_v}$ dominant compared to $\boldsymbol{g_\theta}$.

Fast decoupled Newton method:

- system of $2N$ equations decomposed into two systems of N equations
- each Newton iteration decomposed into : one half-iteration to update the phase angles, followed by one half-iteration to update the voltage magnitudes
- active power and phase reference ($\theta_N = 0$) equations solved using the sub-matrix $\left[\begin{array}{c} \mathbf{f}_\theta \\ \boldsymbol{e_N} \end{array}\right]$ to update the voltage phase angles
- reactive power and voltage equations solved using the sub-matrix $\left[\begin{array}{c} \mathbf{g}_v \\ \boldsymbol{U} \end{array}\right]$ to update the voltage magnitudes.

The "DC" power flow approximation

Simplified power flow equations obtained after:

- linearizing the variation of active power with voltage phase angles
- neglecting the active power losses in all branches
- assuming all voltage magnitudes equal to 1 pu
- neglecting all reactive power flows in branches.

Linear model used:

- to simplify some computations, e.g.
  - to perform a very large number of power flow computations: a single linear system of half size is solved
  - when the power flow equations are included as constraints in a large optimization problem (*Optimal power flow*)
- to easily cumulate the effects of several modifications applied to the system (thanks to linearity).

Approximate linear model

Thus, we assume:

- $V_i = V_j \simeq 1$ pu
- $G_{ij} \simeq 0 \qquad B_{ij} = -\frac{1}{X_{ij}}$ (EHV transmission networks)
- $n_{ij} \simeq 1$ (transformer ratios influence reactive power flows mainly)

Relation between active power injections and voltage phase angles:

$$P_i \simeq \sum_{j \in \mathcal{N}(i)} \frac{1}{X_{ij}} \sin(\theta_i - \theta_j + \phi_{ij}) \simeq \sum_{j \in \mathcal{N}(i)} \frac{\theta_i - \theta_j + \phi_{ij}}{X_{ij}} \tag{1}$$

with the N-th bus as reference: $\theta_N = 0$.

Active power balance with network losses neglected:

$$\sum_{i=1}^{N} P_i = 0 \quad \Leftrightarrow \quad P_N = -\sum_{i=1}^{N-1} P_i$$

With unit voltages and zero reactive power flow, we have in per unit :

$$I_{ij} = |P_{ij}| = \left|\frac{\theta_i - \theta_j + \phi_{ij}}{X_{ij}}\right|$$

Matrix form

Let us assume for simplicity that there is no phase shifting transformer:

$$\phi_{ij} = 0$$

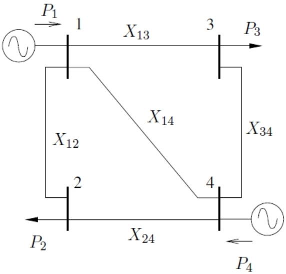
By grouping the equations (1) relative to buses 1 to $N-1$:

$$\boldsymbol{p}^o = \boldsymbol{A}\ \boldsymbol{\theta}$$

where $\boldsymbol{A}$ is defined by:

$$\begin{aligned}
[\boldsymbol{A}]_{ij} &= -\frac{1}{X_{ij}} \qquad i,j = 1,\dots,N-1; \quad i \neq j \\
[\boldsymbol{A}]_{ii} &= \sum_{j \in \mathcal{N}(i)} \frac{1}{X_{ij}} \qquad i = 1,\dots,N-1
\end{aligned}$$

Example

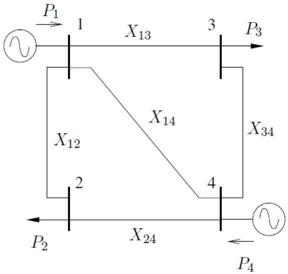


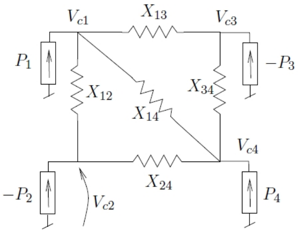
Phase angle reference at bus 4: $\theta_4 = 0$

$$\begin{bmatrix} P_1 \\ -P_2 \\ -P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{X_{12}} + \frac{1}{X_{13}} + \frac{1}{X_{14}} & -\frac{1}{X_{12}} & -\frac{1}{X_{13}} \\ -\frac{1}{X_{12}} & \frac{1}{X_{12}} + \frac{1}{X_{24}} & 0 \\ -\frac{1}{X_{13}} & 0 & \frac{1}{X_{13}} + \frac{1}{X_{34}} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Active power balance: $P_1 - P_2 - P_3 + P_4 = 0$

Why is this called “Direct Current (DC)” ?





resistive circuit with direct current sources

Voltage reference: bus 4 (not the earth in this case !)

Circuit equations obtained from the nodal admittance matrix:

$$\begin{bmatrix} P_1 \\ -P_2 \\ -P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{X_{12}} + \frac{1}{X_{13}} + \frac{1}{X_{14}} & -\frac{1}{X_{12}} & -\frac{1}{X_{13}} \\ -\frac{1}{X_{12}} & \frac{1}{X_{12}} + \frac{1}{X_{24}} & 0 \\ -\frac{1}{X_{13}} & 0 & \frac{1}{X_{13}} + \frac{1}{X_{34}} \end{bmatrix} \begin{bmatrix} V_{c1} - V_{c4} \\ V_{c2} - V_{c4} \\ V_{c3} - V_{c4} \end{bmatrix}$$

By taking $V_{c4} = 0$, we have: $V_{c1} = \theta_1 \quad V_{c2} = \theta_2 \quad V_{c3} = \theta_3$

Sensitivity analysis

Power flow equations in compact vector form:

$$\varphi(\boldsymbol{x}, \boldsymbol{p}) = \mathbf{0} \qquad (2)$$

$\boldsymbol{x}$: vector of voltage magnitudes and phase angles (dim. n) (*state vector*)
$\boldsymbol{p}$: vector of parameters (dim. m)

Let $\boldsymbol{x}^\star$ be the solution of (2) corresponding to $\boldsymbol{p} = \boldsymbol{p}^\star$
$\eta(\boldsymbol{x}, \boldsymbol{p})$ be a variable of interest.

How does η vary when the parameters $\boldsymbol{p}$ are varied from $\boldsymbol{p}^\star$ to $\boldsymbol{p}^\star + \Delta\boldsymbol{p}$?

"Brute-force" solution

Solve (2) for $\boldsymbol{p} = \boldsymbol{p}^\star + \Delta\boldsymbol{p}$, i.e. compute $\Delta\boldsymbol{x}$ such that:

$$\varphi(\boldsymbol{x}^\star + \Delta\boldsymbol{x}, \boldsymbol{p}^\star + \Delta\boldsymbol{p}) = \mathbf{0}$$

and compute the corresponding new value: $\eta(\boldsymbol{x}^\star + \Delta\boldsymbol{x}, \boldsymbol{p}^\star + \Delta\boldsymbol{p})$.

Elegant solution

Compute directly the sensitivities: $S_{\eta \boldsymbol{p}} = \begin{bmatrix} \lim_{\Delta p_1 \to 0} \dfrac{\Delta \eta}{\Delta p_1} \\ \vdots \\ \lim_{\Delta p_m \to 0} \dfrac{\Delta \eta}{\Delta p_m} \end{bmatrix}$

For infinitesimal variations denoted by $d.$:

$$\varphi(\boldsymbol{x}^\star + d\boldsymbol{x}, \boldsymbol{p}^\star + d\boldsymbol{p}) \simeq \varphi(\boldsymbol{x}^\star, \boldsymbol{p}^\star) + \varphi_{\boldsymbol{x}}\, d\boldsymbol{x} + \varphi_{\boldsymbol{p}}\, d\boldsymbol{p} = \varphi_{\boldsymbol{x}}\, d\boldsymbol{x} + \varphi_{\boldsymbol{p}}\, d\boldsymbol{p} = \mathbf{0}$$

and hence, assuming that $\varphi_{\mathbf{x}}$ is non-singular:

$$d\boldsymbol{x} = -\varphi_{\boldsymbol{x}}^{-1}\, \varphi_{\boldsymbol{p}}\, d\boldsymbol{p} \tag{3}$$

By linearizing η :

$$d\eta = \sum_i \frac{\partial \eta}{\partial p_i} dp_i + \sum_i \frac{\partial \eta}{\partial x_i} dx_i = d\mathbf{p}^T \nabla_{\boldsymbol{p}} \eta + d\boldsymbol{x}^T \nabla_{\boldsymbol{x}} \eta$$

$$\text{where} \quad \nabla_{\boldsymbol{p}} \eta = \begin{bmatrix} \frac{\partial \eta}{\partial p_1} \\ \vdots \\ \frac{\partial \eta}{\partial p_m} \end{bmatrix} \qquad \nabla_{\boldsymbol{x}} \eta = \begin{bmatrix} \frac{\partial \eta}{\partial x_1} \\ \vdots \\ \frac{\partial \eta}{\partial x_n} \end{bmatrix}$$

Replacing $d\boldsymbol{x}$ by its expression (3):

$$d\eta = d\boldsymbol{p}^T \nabla_{\boldsymbol{p}}\eta - d\boldsymbol{p}^T \varphi_{\boldsymbol{p}}^T \left(\varphi_{\boldsymbol{x}}^T\right)^{-1} \nabla_{\boldsymbol{x}}\eta = d\mathbf{p}^T \left[\nabla_{\boldsymbol{p}}\eta - \varphi_{\boldsymbol{p}}^T \left(\varphi_{\boldsymbol{x}}^T\right)^{-1} \nabla_{\boldsymbol{x}}\eta\right]$$

which gives the sought sensitivities: $S_{\eta\boldsymbol{p}} = \nabla_{\boldsymbol{p}}\eta - \varphi_{\boldsymbol{p}}^T \left(\varphi_{\boldsymbol{x}}^T\right)^{-1} \nabla_{\boldsymbol{x}}\eta$

Practical procedure

1. compute $\nabla_{\mathbf{x}}\eta$, $\nabla_{\mathbf{p}}\eta$ and $\varphi_{\mathbf{p}}$
2. $\varphi_{\mathbf{x}}$ being available in factorized form (LDU), solve: $\varphi_{\mathbf{x}}^T \, \mathbf{y} = \nabla_{\mathbf{x}}\eta$
3. compute $S_{\eta\mathbf{p}} = \nabla_{\mathbf{p}}\eta - \varphi_{\mathbf{p}}^T \mathbf{y}$.

Examples

Sensitivities to the bus active and reactive powers:

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}^o \\ \boldsymbol{q}^o \end{bmatrix} \quad \text{and} \quad \varphi_{\mathbf{p}} = -\begin{bmatrix} \boldsymbol{U}_p \\ \mathbf{0} \\ \boldsymbol{U}_q \\ \mathbf{0} \end{bmatrix}$$

$\boldsymbol{U}_p$, $\boldsymbol{U}_q$: matrices including 0's and 1's.

Sensitivity of the voltage at the i-th bus

$$\eta = V_i \qquad \nabla_{\boldsymbol{p}}\eta = \mathbf{0} \qquad \nabla_{\boldsymbol{x}}\eta = \boldsymbol{e}_{V_i}$$

$\boldsymbol{e}_{V_i}$: unit vector in which the component equal to 1 corresponds to V_i.

Sensitivity of reactive power produced by the generator at the i-th bus (a PV bus)

$$\eta = Q_{gi}(\boldsymbol{x}) \qquad \nabla_{\boldsymbol{p}}\eta = \mathbf{0}$$

$\nabla_{\boldsymbol{x}}\eta = \nabla_{\boldsymbol{x}} Q_{gi}$: the nonzero partial derivatives correspond to the voltage magnitudes and the phase angles at bus i and at the direct neighbours of bus i.

Sensitivity of active power losses

$$\eta = p = P_N + \sum_{i=1}^{N-1} P_i = P_N(\mathbf{x}) + \sum_{i=1}^{N-1} P_i \qquad \nabla_{\mathbf{p}}\eta = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{1}$: vector of dimension $N-1$ with all components equal to 1
$\mathbf{0}$: zero vector of dimension $N-1$.
$\nabla_{\boldsymbol{x}}\eta = \nabla_{\boldsymbol{x}} P_N$: the nonzero partial derivatives correspond to the voltage magnitudes and the phase angles at bus N and at the direct neighbours of bus N.