

ELEC0014 - Introduction to electric power and energy systems

Powers in the sinusoidal steady state

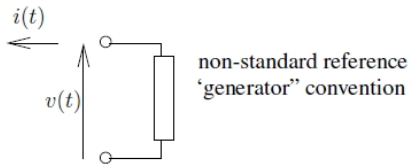
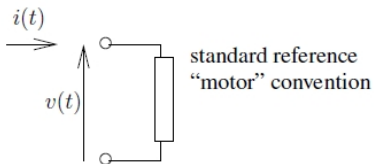
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Power absorbed (or produced) by a one-port



$v(t)$ = voltage *across* the one-port

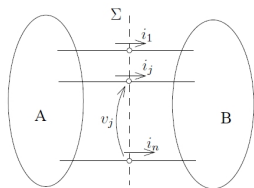
= voltage at the head of the arrow – voltage at the tail of the arrow

$i(t)$ = current *through* the one-port

- *standard reference* in Circuit Theory or *motor convention*:
current counted positive when it enters the one-port by the head of the voltage arrow
 $p(t) = v(t) i(t)$ = instantaneous power **absorbed by** the one-port
- *non standard reference* or *generator convention*:
current counted positive when it leaves the one-port by the head of the voltage arrow
 $p(t) = v(t) i(t)$ = instantaneous power **generated by** the one-port.

Power flowing through a set of conductors

Generalizing the case of the one-port to n conductors ($n \geq 2$).



We assume that the currents are such that:

$$\sum_{j=1}^n i_j = 0 \quad (1)$$

The n -th conductor is taken as voltage reference

- 1 there is no link between A and B other than the n conductors (Σ is a “cut”):
Eq. (1) comes from Kirchhoff's current law
- 2 there is at least one link between A and B:
we assume operation is such that (1) is satisfied.

With the (shown) motor convention for circuit B, $p(t) = \sum_{j=1}^{n-1} v_j i_j$ is the power flowing through the n conductors, from A to B (absorbed by B / produced by A).

Alternating voltage and currents, phasors

$$v(t) = \sqrt{2} V \cos(\omega t + \theta)$$

$$i(t) = \sqrt{2} I \cos(\omega t + \psi)$$

- $\sqrt{2}V, \sqrt{2}I$: amplitudes, or *peak* values
- V, I : *Root Mean Square (RMS)* or *effective* values (mainly used in practice)
- ω : angular frequency = $2\pi f = \frac{2\pi}{T}$

Let us define the *phasors* :

$$\bar{V} = V e^{j\theta}$$

$$\bar{I} = I e^{j\psi}$$

We obviously have:

$$v(t) = \sqrt{2} \operatorname{re} \left(V e^{j(\omega t + \theta)} \right) = \sqrt{2} \operatorname{re} \left(\bar{V} e^{j\omega t} \right)$$

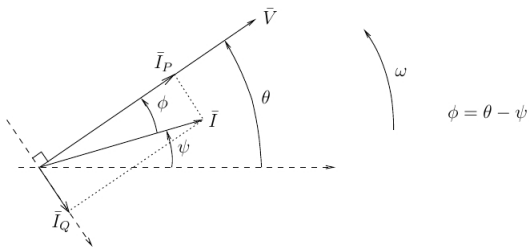
$$i(t) = \sqrt{2} \operatorname{re} \left(I e^{j(\omega t + \psi)} \right) = \sqrt{2} \operatorname{re} \left(\bar{I} e^{j\omega t} \right)$$

In the complex plane, $\bar{V} e^{j\omega t}$ (resp. $\bar{I} e^{j\omega t}$) is associated with a rotating vector:

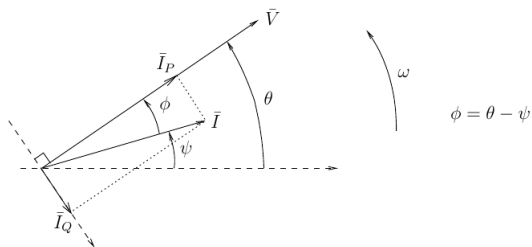
- starting from the origin $0 + j0$
- ending up in this complex number
- whose projection on the real axis is $\frac{1}{\sqrt{2}} v(t)$ (resp. $\frac{1}{\sqrt{2}} i(t)$)

The phasor is the position at $t = 0$ of the rotating vector.

Phasor diagram : graphical representation of the phasors



Active and reactive currents



The angle ϕ starts from the **current** phasor and ends up on **voltage** phasor

Projection of \bar{I} on the axis defined by \bar{V} :

$$\bar{I}_P = I_P e^{j\theta} \quad \text{with} \quad I_P = I \cos(\theta - \psi) = I \cos \phi$$

I_P : a real number, positive if \bar{I}_P has the same direction as $\bar{V} =$ *active current*

Projection of \bar{I} on the axis orthogonal to \bar{V} and *lagging* \bar{V} :

$$\bar{I}_Q = I_Q e^{j(\theta - \frac{\pi}{2})} \quad \text{with} \quad I_Q = I \sin(\theta - \psi) = I \sin \phi$$

I_Q : a real number, positive if \bar{I}_Q is *lagging* $\bar{V} =$ *reactive current*

Instantaneous, active, reactive, fluctuating and apparent powers

$$\begin{aligned}
 i(t) &= \sqrt{2} \operatorname{re} (\bar{I} e^{j\omega t}) = \sqrt{2} \operatorname{re} (\bar{I}_P e^{j\omega t} + \bar{I}_Q e^{j\omega t}) = \sqrt{2} \operatorname{re} (I_P e^{j(\omega t + \theta)} + I_Q e^{j(\omega t + \theta - \frac{\pi}{2})}) \\
 &= \sqrt{2} I_P \cos(\omega t + \theta) + \sqrt{2} I_Q \sin(\omega t + \theta)
 \end{aligned}$$

Instantaneous power:

$$\begin{aligned}
 p(t) = v(t) i(t) &= 2VI_P \cos^2(\omega t + \theta) + 2VI_Q \cos(\omega t + \theta) \sin(\omega t + \theta) \\
 &= VI_P [1 + \cos 2(\omega t + \theta)] + VI_Q \sin 2(\omega t + \theta)
 \end{aligned}$$

Component $VI_P [1 + \cos 2(\omega t + \theta)]$:

- a constant + a component oscillating at the double frequency $2f$
- never changes sign: power always flows in the same direction

Component $VI_Q \sin 2(\omega t + \theta)$:

- zero average \rightarrow corresponds to “no useful work”
- in an RLC circuit, the presence of I_Q is due to the inductors and capacitors
- $VI_Q \sin 2(\omega t + \theta)$ is the time derivative of the energy stored in inductors and capacitors
- this energy is always positive: these elements accumulate and release energy successively. They never release more energy than what they have received!

Average value of $p(t) = VI_P = \text{active (or real) power } P$

$$P = V I_P = V I \cos(\theta - \psi) = V I \cos \phi$$

Amplitude of $VI_Q \sin 2(\omega t + \theta) = VI_Q = \text{reactive power } Q$

$$Q = V I_Q = V I \sin(\theta - \psi) = V I \sin \phi$$

Fluctuating power : sum of components oscillating at frequency $2f$:

$$\begin{aligned} p_f(t) &= VI_P \cos 2(\omega t + \theta) + VI_Q \sin 2(\omega t + \theta) \\ &= VI \cos(\theta - \psi) \cos 2(\omega t + \theta) + VI \sin(\theta - \psi) \sin 2(\omega t + \theta) \\ &= VI \cos(2\omega t + \theta + \psi) \end{aligned}$$

- corresponds to no useful work
- P is the only useful component of the total power $p(t)$

Apparent power: used to dimension equipment

$$S = VI \quad \Rightarrow \quad S = P \quad \text{when} \quad \phi = 0$$

Units

$p(t)$, $p_f(t)$ and P	watt	W	kW, MW
Q	var	(VAr, Var or) var	kvar, Mvar
S	volt.ampere	VA	kVA, MVA

Numerical example

$$\bar{V} = 1 \angle 0 \quad \bar{I} = 0.5 \angle -\frac{\pi}{6} \quad \omega = 2\pi 50 = 314 \text{ rad/s}$$

Hence:

$$v(t) = \sqrt{2} \cos(\omega t) \quad i(t) = \sqrt{2} 0.5 \cos(\omega t - \frac{\pi}{6})$$

$$\phi = 0 - (-\frac{\pi}{6}) = \frac{\pi}{6}$$

$$I_P = 0.5 \cos \frac{\pi}{6} = 0.433$$

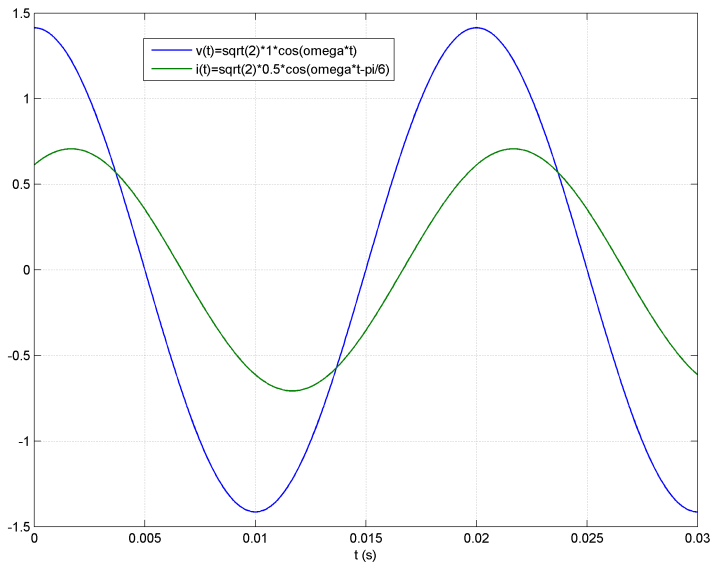
$$I_Q = 0.5 \sin \frac{\pi}{6} = 0.25$$

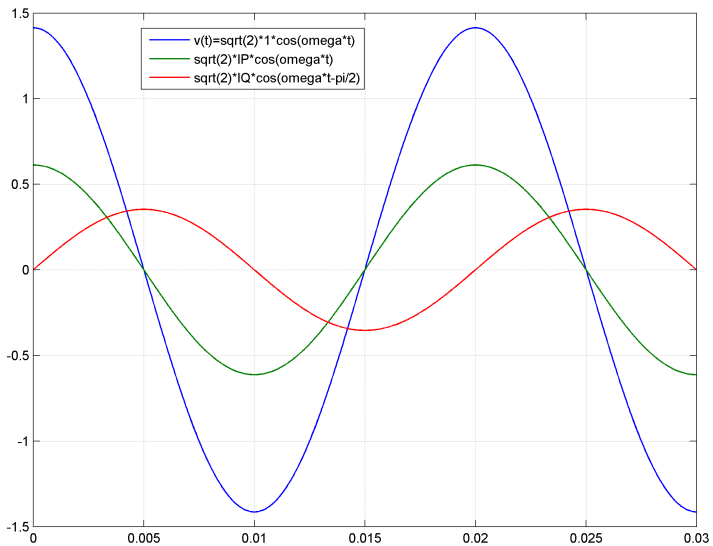
$$I = \sqrt{I_P^2 + I_Q^2} = 0.5$$

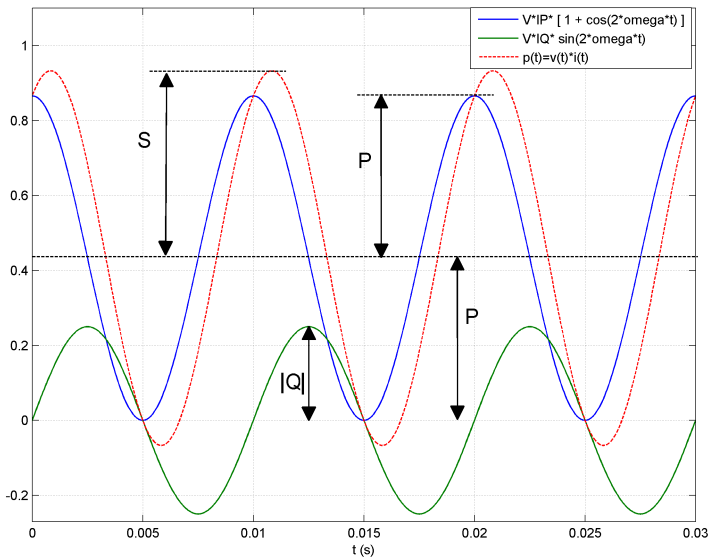
$$P = VI_P = 0.433$$

$$Q = VI_Q = 0.25$$

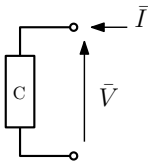
$$S = VI = 0.5$$





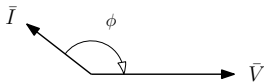


Consumer or producer ?

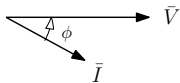


$P = VI \cos \phi$ is the active power **consumed** by C

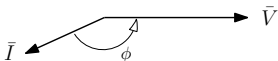
$Q = VI \sin \phi$ is the reactive power **consumed** by C



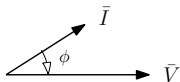
C produces active power
C produces reactive power



C consumes active power
C consumes reactive power

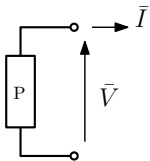


C produces active power
C consumes reactive power



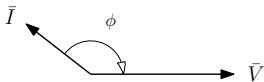
C consumes active power
C produces reactive power

Consumer or producer ?

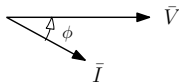


$P = VI \cos \phi$ is the active power **produced by P**

$Q = VI \sin \phi$ is the reactive power **produced by P**



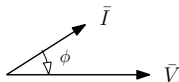
P consumes active power
consumes reactive power



P produces active power
produces reactive power



P consumes active power
produces reactive power



P produces active power
consumes reactive power

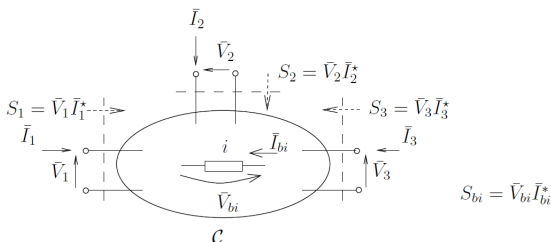
Complex power

$$\begin{aligned} S &= \bar{V} \bar{I}^* \\ &= V e^{j\theta} I e^{-j\psi} = VI e^{j(\theta-\psi)} \\ &= VI \cos(\theta - \psi) + jVI \sin(\theta - \psi) = P + jQ \\ \Rightarrow \operatorname{re} S &= P \\ \operatorname{im} S &= Q \\ |S| &= \sqrt{P^2 + Q^2} = VI \end{aligned}$$

Theorem of conservation of complex power

in a circuit fed by sinusoidal sources of the same frequency, the sum of complex powers entering any part of the circuit is equal to the sum of the complex powers received by all branches of this part of the circuit.

Example



$$S_1 + S_2 + S_3 = \sum_i S_{bi} \quad \Rightarrow \quad P_1 + P_2 + P_3 = \sum_i P_{bi} \quad \text{and} \quad Q_1 + Q_2 + Q_3 = \sum_i Q_{bi}$$

Conservation of power:

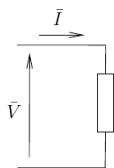
- obvious for instantaneous power (time derivative of energy)
- natural for active power (average of power)
- remarkable for reactive power !

Application to electric power systems :

power injected in network =
consumption of all loads + losses in all network elements

Expressions relative to one-ports

$\bar{V} = Z \bar{I} = (R + jX) \bar{I}$ Z : impedance R : resistance X : reactance	$\bar{I} = Y \bar{V} = (G + jB) \bar{V}$ Y : admittance G : conductance B : susceptance
$S = Z I^2$ $P = R I^2$ $Q = X I^2$	$S = Y^* V^2$ $P = G V^2$ $Q = -B V^2$



	powers consumed by a		
	resistance R	inductance L	capacitance C
ϕ	0	$\pi/2$	$-\pi/2$
P	$R I^2 = \frac{V^2}{R}$	0	0
Q	0	$\omega L I^2 = \frac{V^2}{\omega L}$	$-\frac{I^2}{\omega C} = -\omega C V^2$

An inductor **consumes** reactive power. A capacitor **produces** reactive power.

Power factor

$\cos \phi$ is called the *power factor*

- the larger the phase angle between the voltage and the current phasors, the smaller the power factor

$$I = \frac{P}{V \cos \phi}$$

- for the same useful power P and under a constant voltage V , the smaller $\cos \phi$, the larger the current I

$$I = \frac{\sqrt{P^2 + Q^2}}{V}$$

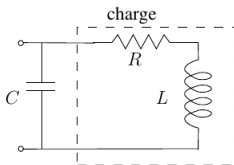
- for the same useful power P and under a constant voltage V , the larger the reactive power Q **consumed or produced** by the load, the larger the current I

A larger current :

- requires larger sections of conductors \Rightarrow more costly investment !
- causes larger Joule losses RI^2 in those conductors \Rightarrow more costly operation !

Compensation of inductive loads

Produce reactive power to bring the power factor closer to one.



$$\cos \phi = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{RI^2}{\sqrt{R^2I^4 + \omega^2L^2I^4}} = \frac{R}{\sqrt{R^2 + \omega^2L^2}}$$

Ideal compensation: Q_c produced by capacitor = Q_ℓ consumed by load:

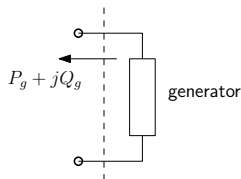
$$\omega CV^2 = \frac{\omega L V^2}{R^2 + \omega^2 L^2} \Leftrightarrow C = \frac{L}{R^2 + \omega^2 L^2}$$

- Compensation must be adjusted with the time variations of load.
- For large industrial loads with fast variations of P or Q , the insertion/removal of capacitors by switching breakers on/off has to be replaced by faster power electronics-based devices.
- Do not overcompensate; it also makes harm !

What about generators ?

The power factor can be also defined for generators:

$$\cos \phi = \frac{P_g}{\sqrt{P_g^2 + Q_g^2}}$$





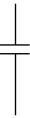
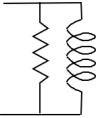

- a generator can produce or absorb reactive power
- whether it absorbs or consumes is not shown by the power factor
- hence, sometimes the value of the power factor is followed by
 - “inductive” if reactive power is produced (the gener. feeds an inductive load)
 - “capacitive” if reactive power is consumed (the gener. feeds a capacitive load)
- less ambiguous:

$$\tan \phi = \frac{Q_g}{P_g}$$

A small quiz

Fill the cells of the table below with the most appropriate answer among :

$= 0$ < 0 > 0 $= 1$ < 1

one-port :					
active power consumed					
reactive power produced					
$\cos \phi$					
$\tan \phi$ ¹					

¹assume motor convention

Exercises

If $a(t)$ and $b(t)$ have sinusoidal evolutions:

$$\begin{aligned} a(t) &= \sqrt{2}A \cos(\omega t + \theta_A) && \text{with phasor } \bar{A} = A e^{j\theta_A} \\ b(t) &= \sqrt{2}B \cos(\omega t + \theta_B) && \text{with phasor } \bar{B} = B e^{j\theta_B} \end{aligned}$$

show that: $\langle a(t)b(t) \rangle = \frac{1}{T} \int_0^T a(u)b(u)du = \text{real}(\bar{A}\bar{B}^*)$ where $T = \frac{2\pi}{\omega}$

Use this property to show that in the sinusoidal steady state:

- the active power is the average instantaneous power: $P = \langle v(t)i(t) \rangle$
- the reactive power is given by: $Q = \langle v(t - \frac{T}{4})i(t) \rangle = \langle v(t)i(t + \frac{T}{4}) \rangle$
- the reactive power consumed by a(n ideal) coil relates to the average magnetic energy stored in that coil through: $Q = 2\omega \langle W_m \rangle$
- the reactive power produced by a(n ideal) capacitor relates to the average electric energy stored in that capacitor through: $Q = 2\omega \langle W_e \rangle$